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Cluster formation in a Fermi system with long-range attraction

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Abstract

Based on a statistical approach we describe the possible formation of a spatially inhomogeneous distribution in the system of interacting Fermi particles by long-range forces, and we demonstrate the nonperturbative calculation of the partition function in this case. It is shown that particles interacting with an attractive $1/r$ potential form clusters when the pressure due to the interactions balances the effective repulsion due to the Fermi statistics. A cluster is the equilibrium structure if we suppose that the average energy of interaction of two particles is much less than their average kinetic energy. The dynamics of cluster formation is considered in this approach and the time of relaxation to the equilibrium state is found. It is shown that phase transition from a spatially inhomogeneous state to a homogeneous state only occurs in a finite system. The temperature of such a phase transition is determined by the size of the finite system and the average density.

1. Introduction

The formation of a spatially inhomogeneous distribution of interacting particles is a typical problem in condensed matter physics. The types of spatial structures and conditions for their formation are determined by the type of interaction. A cluster is one form of spatial inhomogeneity. For a gas with Coulomb type attraction between particles (a self-acting system) we cannot calculate the virial coefficients (we cannot do it for the interaction $1/r^n$ if $n \leq 3$ [27]). The statistical description of such a system developed in [1–5] is based on the application of quantum field theory [12–17, 23]. In these works it was shown that gas with interacting particles is equivalent to two scalar fields corresponding to attraction and repulsion, respectively, with an exponential self-action. The partition function can be represented in terms of a functional integral over these auxiliary fields. The extremal conditions for this functional are nonlinear equations. The spatial distribution function which describes the cluster is the soliton solution of the equations. This approach gives the possibility of finding the spatial distribution of particles, calculating the cluster size and determining the temperature of the phase transition in to the state under consideration by nonperturbative calculation.

Constancy of the chemical potential in space is a condition for absence of evaporation or swelling of the cluster.

Mechanical equilibrium results from equality of pressures on the cluster's borders. Taking into consideration the particles outside the cluster (i.e. the characteristics of the interacting gas), we determine the parameters of the cluster and the equation of state. In previous papers the system of interacting particles was considered in the limit $N \rightarrow \infty$, $V \rightarrow \infty$, but N/V is fixed. However, it is known that real systems have a finite volume with a finite number of particles in it. This fact must be taken into account in calculation of the equation of state, but necessitates a small correction for systems with short-range interaction between particles [20]. In the case of systems with long-range interaction (for example gravitation) the situation must be changed drastically, because the length of such an interaction is equal to or bigger than the size of the system. In particular, this must have an effect on the process of cluster formation.

In the papers [6, 7] this approach was applied to the investigation of a system with long-range interacting Bose particles. It has been shown that the spatial distribution of such a system is inhomogeneous and is like a finite-size cluster. The radius of the cluster and the conditions and dynamics of its formation have been found both for nondegenerate states and for the Bose–Einstein condensation regime. In this article, based on a statistical approach [1–4], we demonstrate the nonperturbative calculation of the partition function, we solve

the system of Fermi particles interacting with an attractive $1/r$ potential, that is we obtain free energy for such a system and consider phase transition between a spatially homogeneous state and a spatially inhomogeneous state. We obtain the expression for the equilibrium radius of a cluster and the conditions for formation of a spatially inhomogeneous distribution in such a Fermi system for all temperatures. However, we can find exact results in the Boltzmann limit with first quantum corrections. For all temperatures we propose an approach which gives the possibility of finding the approximate radius of a cluster or of evaluating it.

2. Statistical approach to the system of interacting particles

Let us consider a system of interacting particles in conditions such that the thermal wavelength of a particle can be larger than the average interparticle distance: on the one hand it is necessary to take into account statistical factors, but, on the other hand, this length is far smaller than the average scattering length, allowing us to describe the interaction classically, disregarding dynamical quantum correlations. The Hamiltonian of such a system [1, 3, 17, 18] is

$$H(n) = \sum_s \varepsilon_s n_s - \frac{1}{2} \sum_{ss'} W_{ss'} n_s n_{s'} + \frac{1}{2} \sum_{ss'} U_{ss'} n_s n_{s'}, \quad (1)$$

where ε_s is the additive part of the particle energy in the state s (for example kinetic energy or energy in an external field) and $W_{ss'}$ and $U_{ss'}$ are the absolute values of the attraction and repulsion energies of particles in the states s and s' , respectively. The macroscopic state of the system is determined by the occupation numbers n_s . The subscript s corresponds to variables that describe an individual particle state.

In [1–4] in order to investigate the thermodynamic properties of the system of interacting particles a Hubbard–Stratonovich [8–11] representation was used for the partition function in the terms of a grand partition function:

$$Z_n = \frac{1}{2\pi i} \oint d\xi \int D\varphi \int D\psi \exp(-S(\xi, \varphi, \psi)), \quad (2)$$

where S is the functional which we call an effective free energy, analogous to an action in field theory:

$$\begin{aligned} S(\xi, \varphi, \psi) = & \frac{1}{2\beta} \sum_{s,s'} (W_{s,s'}^{-1} \varphi_s \varphi_{s'} + U_{s,s'}^{-1} \psi_s \psi_{s'}) \\ & + \delta \sum_s \ln(1 - \delta\xi \exp(-\beta\varepsilon_s + \varphi_s) \cos \psi_s) \\ & + (N + 1) \ln \xi, \end{aligned} \quad (3)$$

$\xi \equiv e^{\beta\mu}$ is activity, μ is chemical potential, $\beta = 1/kT$ is reverse temperature, s and s' run all the states of the system, ε is kinetic energy, N is the number of particles, $\delta = +1$ for Bose particles and -1 for Fermi particles. The two auxiliary fields φ and ψ are introduced corresponding to attraction and repulsion. The partition function (2) is written as a functional integral over these fields. $W_{s,s'}^{-1}$, $U_{s,s'}^{-1}$ are inverse operators of

the interaction: $\omega_{ss'}^{-1} = \delta_{ss'} \hat{L}_{s'}$ where $\hat{L}_{s'}$ is such an operator for which the interaction potential is a Green function.

The integral (2) is calculated by the ‘saddle-point’ method [19, 20] across the point determined by the functional derivatives $\frac{\delta S}{\delta \varphi} = \frac{\delta S}{\delta \psi} = 0$ as

$$\frac{1}{\beta} \sum_{s'} W_{ss'}^{-1} \varphi_{s'} - \frac{\xi_s e^{\varphi_s} \cos \psi_s}{1 - \delta \xi_s e^{\varphi_s} \cos \psi_s} = 0 \quad (4)$$

$$\frac{1}{\beta} \sum_{s'} U_{ss'}^{-1} \psi_{s'} + \frac{\xi_s e^{\varphi_s} \sin \psi_s}{1 - \delta \xi_s e^{\varphi_s} \cos \psi_s} = 0 \quad (5)$$

and the derivative

$$\frac{\partial S}{\partial \xi} = 0 \Rightarrow \sum_s \frac{\xi_s e^{\varphi_s} \cos \psi_s}{1 - \delta \xi_s e^{\varphi_s} \cos \psi_s} = N + 1, \quad (6)$$

where $\xi_s = \xi e^{\beta\varepsilon_s}$. This set of equations (4)–(6) provides a solution to the many-particle problem in the sense that it selects the system states whose contributions to the partition function are dominant. The third equation is a normalization condition for the particle distribution function which is determined by the auxiliary fields. It is obvious that, for given statistics, the distribution function depends on the nature and intensity of the interaction. Cluster formation corresponds to particle localization within a limited space. In our treatment, the effect is reflected in the behavior of the auxiliary fields and chemical potential.

For a screened Coulomb or Newtonian potential, the inverse operator may be written as

$$\hat{L} = -\frac{1}{4\pi q^2} (\Delta - \lambda^2), \quad (7)$$

where q^2 is the interaction constant, Δ is Laplace’s operator and λ^{-1} is the screening length [8, 15–17]. The number of realistic interactions for which the inverse operator can be found and the general solution of the set (4)–(6) can be obtained is limited. That is why we will confine ourselves to long-range attractive and repulsive Coulomb potentials ($\propto \frac{1}{R}$) for a Fermi gas where we will find the conditions for the cluster formation and cluster parameters.

In the continuum approximation, the subscript s runs through a continuum of values in the system of volume V . When integrating over impulses and coordinates, we bear in mind that the unit cell volume in the space of individual states is equal to $\omega = (2\pi\hbar)^3$. In order to avoid unnecessary complications we shall only consider particles without spin.

3. Cluster formation in the system of interacting particles

3.1. The equations for the spatial distribution function for a system with Coulomb attraction

Now we shall consider the system of particles interacting by long-range attraction only and we shall demonstrate a nonperturbative calculation of the partition function. For Coulomb attraction the inverse operator is known to be

$W_{rr'}^{-1} = \frac{-1}{4\pi q^2} \Delta_r \delta_{rr'}$, where g is the interaction constant and Δ_r is the Laplace operator.

Let us consider the system in the usual thermodynamic limit: number of particles $N \rightarrow \infty$ and volume $V \rightarrow \infty$ with fixed N/V . We consider the effective free energy (3) in spherical coordinates: $\nabla \equiv \frac{\partial}{\partial R}$ and, neglecting the surface contribution, we can write the effective free energy (3) for a Fermi gas as

$$S = \frac{1}{2\beta} \int_0^V \frac{(\nabla\varphi)^2}{4\pi q^2} dV - \frac{1}{\omega} \int_0^V dV \times \int d^3 p \ln \left[1 + \xi e^\varphi \exp\left(-\beta \frac{p^2}{2m}\right) \right] + N \ln \xi = \int_0^V dV \left[\frac{(\nabla\varphi)^2}{4r_q} - \frac{1}{\lambda^3} f_{5/2}(\xi e^\varphi) \right] + N \ln \xi, \quad (8)$$

where $r_q = 2\pi q^2 \beta$, m is a particle mass, $\lambda = \sqrt{\frac{\beta \hbar^2}{2\pi m}}$ is the thermal wavelength of a particle and $f_{5/2}(\xi) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \ln(1 + \xi e^{-x^2}) = \sum_{l=1}^\infty (-1)^{l+1} \frac{\xi^l}{l^{5/2}}$ is a special Fermi function [19, 20]. The sense of the auxiliary field φ is next—a spatial distribution function can be expressed with φ as

$$\rho(R) = \int_0^\infty \frac{d^3 p}{(2\pi \hbar)^3} \frac{\xi \exp\left(-\frac{p^2}{2mkT}\right) e^{\varphi(R)}}{1 + \xi \exp\left(-\frac{p^2}{2mkT}\right) e^{\varphi(R)}}, \quad (9)$$

where we used the underintegral expression in equation (6). Let us introduce the dimensionless quantity $r = R/r_q$ instead of R . Then, the effective free energy (8) (in spherical coordinates) can be written as

$$S = 4\pi \int_0^V r^2 dr \left[\frac{(\nabla\varphi)^2}{4} - \frac{r_q^3}{\lambda^3} f_{5/2}(\xi e^\varphi) \right] + N \ln \xi. \quad (10)$$

The saddle-point equation is a Lagrange equation for this functional:

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{2}{r} \frac{\partial \varphi}{\partial r} + \frac{2r_q^3}{\lambda^3} \frac{\partial f_{5/2}(\xi e^\varphi)}{\partial \varphi} = 0. \quad (11)$$

This equation selects the system's states (which are described by the field configuration $\varphi(R)$ or spatial distribution function $\rho[\varphi(R)]$) whose contributions to the partition function are dominant. Unfortunately, this equation does not have an analytical solution. But the problem simplifies in the Boltzmann limit.

3.2. The solution in the Boltzmann limit and the dynamics of cluster formation

Let us consider the limiting case $\xi \rightarrow 0$ corresponding to high temperature and small concentration, using decomposition of the special function $f_{5/2}(\xi e^\varphi)$ in a row on orders of the activity ξ . Then the effective free energy (8) is reduced to the simpler expression

$$S = 4\pi \int_0^V \left[\frac{1}{4} (\nabla\varphi)^2 - \xi \frac{r_q^3}{\lambda^3} e^\varphi \right] r^2 dr + N \ln \xi. \quad (12)$$

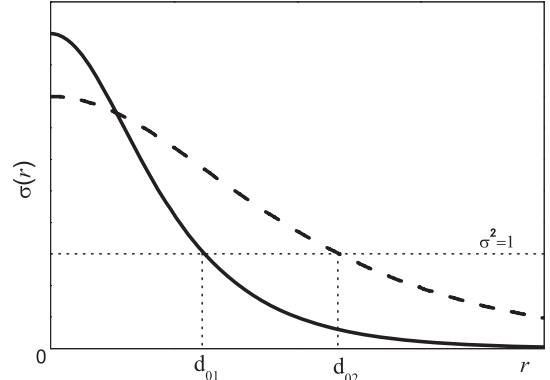


Figure 1. The spatial distribution function $\sigma(r)^2$ in a cluster at different temperatures (schematically). The solid line corresponds to a lower temperature T_1 , the dashed line corresponds to a higher temperature T_2 . The dotted line $\sigma^2 = 1$ corresponds to the mean fixed density of the system and determines the equilibrium radii of clusters d_{01} and d_{02} under the above-mentioned thermodynamic conditions.

That is, the system with Coulomb attraction is equivalent to a single scalar field $\varphi(r)$ with an exponential self-action. An analogous expression was obtained in [23]; however, the term $N \ln \xi$, which fixes the number of particles, is absent there. The saddle-point equation for the functional (12) is

$$\frac{\partial^2 \varphi}{\partial r^2} + \xi \frac{2r_q^3}{\lambda^3} e^\varphi = 0. \quad (13)$$

As will be shown below, the term $\frac{2}{r} \frac{\partial \varphi}{\partial r}$ can be omitted. In order to connect the auxiliary field φ with the density $\rho(R)$ we have to use equation (9) and to pass to the Boltzmann limit $\xi \rightarrow 0$:

$$\rho = \frac{\xi}{\lambda^3} e^\varphi \equiv \frac{\xi}{\lambda^3} \sigma^2, \quad (14)$$

where we introduce the new variable $\sigma = \exp(\varphi/2)$ and mark in $\alpha^2 \equiv r_q^3/\lambda^3$. Then we can rewrite equation (13) for the field φ as an equation for the density as

$$\frac{\partial^2 \sigma}{\partial r^2} - \frac{1}{\sigma} \left(\frac{\partial \sigma}{\partial r} \right)^2 + \xi \alpha^2 \sigma^3 = 0. \quad (15)$$

This equation has a soliton solution [3]

$$\sigma = \frac{\Delta}{\sqrt{\xi \alpha} \cosh \Delta r}, \quad (16)$$

where Δ is an integration constant. Any soliton solution corresponds to a spatially inhomogeneous distribution of particles—a finite-size cluster. The corresponding asymptotics are $\sigma^2 = 1$ for $r = d$, where d is the cluster size, and $\sigma \rightarrow 0$ as $r \rightarrow \infty$. This solution describes the presence of particles in the inhomogeneous formation of size d and the absence of particles at infinity (figure 1). This spatial distribution is compressing to the line $\sigma = 1$ or $\varphi = 0$ when $T \rightarrow \infty$, $\frac{N}{V} \rightarrow 0$, which is why the field $\varphi = 0$ corresponds to a spatially homogeneous distribution in the system.

Let us substitute solution (16) into the effective free energy (12):

$$S = 4\pi \int_0^d (\Delta^2 - 2\xi\alpha^2\sigma^2) r^2 dr + N \ln \xi. \quad (17)$$

Then we will integrate using the decomposition $1/\cosh x \approx 1 - x^2/2$ in power series of $x \equiv \Delta d \ll 1$:

$$S = -V \frac{\Delta^2}{\alpha^2 \lambda^3} + N \ln \xi. \quad (18)$$

The integration constant Δ^2 is found from the asymptotics $\sigma^2(d) = 1$: $\Delta^2 \approx \xi\alpha^2 + \xi^2 d^2 \alpha^4$. Thus, we have the result:

$$S = -\frac{V}{\lambda^3} \xi + N \ln \xi - \frac{V}{\lambda^3} \xi^2 d^2 \alpha^2. \quad (19)$$

Assuming that the average energy of interaction of two particles is less than the average kinetic energy $\sim kT$ of a particle $\frac{r_q^3 N}{V} \ll 1$, we can find the activity ξ from the equation $\frac{\partial S}{\partial \xi} = 0$ as

$$\xi \approx \frac{\lambda^3 N}{V} - \frac{2d^2 \alpha^2 \lambda^6 N^2}{V^2} \equiv \xi_0 + \xi_q, \quad |\xi_q| \ll \xi_0, \quad (20)$$

where ξ_0 and ξ_q are activities of an ideal gas and the first correction for the interaction, respectively. Then integrating equation (2) with the effective free energy (19) on the saddle-point (20) we obtain the partition function as

$$Z_N = Z_N^0 \times \exp \left[\frac{V}{\lambda^3} \xi_q - N \ln \left(1 + \frac{\xi_q}{\xi_0} \right) + \frac{V}{\lambda^3} \xi^2 d^2 \alpha^2 \right], \quad (21)$$

where Z_N^0 is the partition function for an ideal gas. Knowing this, we can find a decomposition of free energy on the cluster radius d which plays a part in the correlation length

$$\begin{aligned} F &= F_0 - kT \left[\frac{V}{\lambda^3} \xi_q - N \ln \left(1 + \frac{\xi_q}{\xi_0} \right) + \frac{V}{\lambda^3} \xi^2 d^2 \alpha^2 \right] \\ &= F_0 + kT \left[-\frac{\alpha^2 \lambda^3 N^2}{V} d^2 + \frac{\alpha^4 \lambda^6 N^3}{V^2} d^4 \right], \end{aligned} \quad (22)$$

where F_0 is the free energy for an ideal gas. We can see that the d^2 term is always negative $-\frac{\alpha^2 \lambda^3 N^2}{V} < 0$, i.e. the cluster exists in a system with Coulomb attraction at any thermodynamic conditions. Minimizing (22) by the size of the cluster

$$\frac{\partial F}{\partial d} = -kT \frac{d\alpha^2 \lambda^3 2N^2}{V} \left[1 - 4 \frac{d^2 \alpha^2 \lambda^3 N}{V} \right] = 0, \quad (23)$$

we obtain the optimum radius of the cluster,

$$d_0^2 = \frac{V}{4Nr_q^3} \quad (24)$$

or, for the dimension value $D = d \cdot r_q$,

$$D_0^2 = \frac{1}{4} \frac{kT}{2\pi q^2} \frac{V}{N}. \quad (25)$$

This expression means that the equilibrium size of the cluster is defined by the balance of the two forces. The first

is the attraction Coulomb energy, which tries to compress the gas. It is represented by the multiplier q^2 (in the sense of an interaction constant). The decrease in cluster size with an increase in the average density in the system N/V is connected with a closer packing of the particles in the cluster on account of the increase in the energy of interaction. The second force is the thermal energy, which creates a positive pressure resisting compression. It is represented by the multiplier kT . With increasing temperature the size of the cluster d is increasing; however, the central density is decreasing. When $T \rightarrow \infty$ the radius of the cluster $d \rightarrow \infty$ and the central density $\sigma(0) \rightarrow 1$. Such behavior of the value d tells us that d plays the part of correlation length, which at the point of phase transition (in our case $T \rightarrow \infty$) is infinitely large. Such a situation is realized due to the long-range type ($\sim 1/r$) of Coulomb interaction.

On other hand equation (25) may be understood as such characteristic distance in the system at which the essential deflection from the average fixed density N/V is observed. If we assume that $q^2 = Gm^2$, where m is particle mass and G is the gravitational constant, this expression may be understood as Jeans' length [22].

Let us consider the dynamics of cluster formation. For this we will use the equation of motion as

$$\frac{\partial d}{\partial t} = -\chi \frac{\partial F}{\partial d}, \quad (26)$$

where χ is some constant. Applying (22) and (24) we have:

$$\dot{d} + \eta d^3 - \eta d_0^2 d = 0, \quad (27)$$

where we use $\eta \equiv \frac{\chi N k T}{2d_0^4}$. The solution of equation (27) is

$$d^2 = \frac{d_0^2}{1 + C \exp(-2\eta d_0^2 t)}, \quad (28)$$

where C is an integration constant. Let us consider the next initial states:

- Let the radius of spatial inhomogeneity be larger than the equilibrium size $d(t) > d_0$ at some moment of time $t > 0$. Then the constant $-1 < C < 0$, which means that the cluster increases from a spatially homogeneous distribution (where $d \rightarrow \infty$) to the equilibrium size d_0 .
- Let the radius of a spatial inhomogeneity be smaller than the equilibrium size $d(t) < d_0$. Then the constant $C > 0$, which means that the cluster will evaporate from an initial distribution with central density which is larger then equilibrium density at the given temperature.

We can see that size of the cluster approaches the equilibrium size (24) asymptotically $d = d_0 [1 + C \exp(-2\eta d_0^2 t)]$ when deflections are small. The term $1/2\eta d_0^2$ can be understood as the relaxation time. If some fluctuation of density has appeared in the system, then it brings about the appearance of a potential gradient. In turn, this brings about spatial inhomogeneity—a cluster with size approaching the equilibrium value asymptotically. In other words, a cluster is the equilibrium structure if we are assuming that the average energy of interaction of two particles is less than the average

kinetic energy $\sim kT$ of a particle $\frac{r_q^3 N}{V} \ll 1$, when the term $\frac{2}{r} \frac{\partial \varphi}{\partial r}$ in equation (13) can be omitted, as it will be in the next subsection.

Since the number of particles in the system $N \rightarrow \infty$, but the number of the particles in the cluster is finite, this means that our system disintegrates to an infinite multitude of clusters of size D_0 each. After that the process of cluster formation is repeated, where the early formed clusters play the part of particles. In others words, the free energy of such a system has no absolute minimum and each state of the system is analogous to a false vacuum in field theory [12].

3.2.1. Single-particle motion. Let us consider single-particle motion in order to ground our assumptions in the previous section for the calculation of cluster size. For the evaluation of cluster size we can make use of another method without calculating the free energy. Let us assume that the average energy of Coulomb interaction of two particles is less than the average kinetic energy $\sim kT$ of a particle, when $\frac{r_q^3 N}{V} \ll 1$ this has been called ‘local ideality approach’ [24]. This must lead to the two consequences:

- the activity must be

$$\xi = \xi_0 \left(1 + O \left[\frac{r_q^3}{V/N} \right] \right), \quad (29)$$

where $\xi_0 = \frac{\lambda^3 N}{V}$ is the activity for an ideal gas. That is the system’s activity differs little from the activity for an ideal gas.

- The center density (figure 1) differs little from the average density: $\rho(0) \simeq mN/V$. Then we can think that $\Delta^2 \simeq \frac{N}{V} r_q^3$ and we can evaluate the term $\frac{2}{r} \frac{\partial \varphi}{\partial r}$ or $\frac{2}{r} \frac{\partial \sigma}{\partial r}$ in equations (13), (15) with the help of solution (16):

$$\frac{\max \left| \frac{2}{r} \frac{\partial \sigma}{\partial r} \right|}{\frac{\partial^2 \sigma}{\partial r^2} - \frac{1}{\sigma} \frac{\partial \sigma}{\partial r} + \xi \frac{r_q^3}{\lambda^3} \sigma^3} \sim \frac{r_q^3 N}{V} \ll 1.$$

That is, this term can be omitted in the local ideality approach.

After that let us calculate the potential energy of a particle $U(r)$ in the cluster using the Boltzmann distribution

$$\sigma(r) = \sigma(0) \exp \left[-\frac{U(r)}{kT} \right]. \quad (30)$$

Then, using (16), we have that

$$U(r) = -kT \ln \left[\frac{\sigma(r)}{\sigma(0)} \right] = 2kT \ln [\cosh (\Delta r)]. \quad (31)$$

The field $U(r)$ is the self-consistent field of the rest particles in the system (in the cluster). Using the assumption (29) about interaction and thermal energies and considering small distances from the center of the cluster as $r/r_q \leq 1$, we can write that

$$U(r) \approx kT \Delta^2 r^2 \approx \frac{kT N r_q^3}{V} r^2. \quad (32)$$

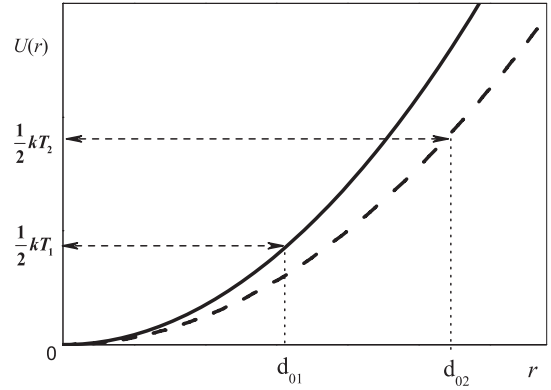


Figure 2. The potential energy $U(r)$ of a particle in the cluster (schematically). The solid line corresponds to a lower temperature T_1 , the dashed line corresponds to a higher temperature T_2 . The particle has finite motion in the potential well created by all the particles in the system. Then we determine the radius of the cluster d as the distance from the center of the cluster to the point at which a particle with average kinetic energy $\frac{1}{2}kT$ stops.

We can see that a particle in the cluster is in a potential well created by all the cluster’s particles. Then the cluster radius d can be understood as a distance from the center of the cluster that

$$U(d) = \frac{1}{2}kT, \quad (33)$$

where the multiplier $1/2$ corresponds to radial degree of freedom. This means that d is the distance from the center of the cluster to the point at which a particle with average kinetic energy $\frac{1}{2}kT$ will stop (figure 2). That is, we can calculate the radius of the cluster in a self-consistent manner.

Then solving the equation (33) we has the equilibrium radius of the cluster as $d^2 = \frac{V}{2r_q^3 N}$, which exactly coincides with equation (24) by a constant multiplier. Since two particles are attracted by a low $1/r$, by Newton mechanics they cannot fall on each other. Hence the spatial distribution is regular everywhere in the limits of the local ideality approach; however, this is not so in the total case [25]. We call such a mechanism of cluster formation the thermal mechanism. It is necessary to notice that single-particle motion in a cluster is superdiffusive [26].

3.3. The cluster size at all temperatures

In order to calculate the radius of a cluster at any temperature we must solve equation (11) and obtain a spatial distribution function of type (16). We cannot do this because of mathematical difficulties; however, if we suppose that the gas is ideal locally, that is $\frac{r_q^3 N}{V} \ll 1$, and consider small distances from the center of the coordinates $R/r_q \leq 1$, where $|\varphi| \ll 1$ (see equations (14) and (16)), then our problem simplifies.

For example, let us consider equation (13) and decompose e^φ in a row:

$$\frac{\partial^2 \varphi}{\partial r^2} + \xi_0 \frac{2r_q^3}{\lambda^3} + O(\varphi) = 0, \quad (34)$$

where we suppose that $\xi \approx \xi_0$. Its solution and the corresponding density is

$$\varphi \approx -\xi_0 \frac{2r_q^3}{\lambda^3} r^2 \Rightarrow \rho = \frac{\xi}{\lambda^3} \exp\left(-\frac{\xi_0 r_q^3}{\lambda^3} r^2\right). \quad (35)$$

Then we can obtain the potential energy of a particle in the cluster as

$$U(r) = -kT \ln \left[\exp\left(-\frac{\xi_0 r_q^3}{\lambda^3} r^2\right) \right] = kT \frac{\xi_0 r_q^3}{\lambda^3} r^2, \quad (36)$$

which was obtained earlier (32). Then, using the condition $U(d) = \frac{1}{2}kT$ and an expression for the Boltzmann activity $\xi_0 = \frac{\lambda^3 N}{V}$ we obtain the result

$$kT \frac{\xi_0 r_q^3}{\lambda^3} d^2 = \frac{1}{2}kT \Rightarrow d^2 = \frac{V}{2r_q^3 N}, \quad (37)$$

which coincides with equation (24).

Now let us calculate the radius of the cluster for any temperature using equation (11) and decomposing one in a row as

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{2r_q^3}{\lambda^3} \left(\frac{\partial f_{5/2}(\xi_0 e^\varphi)}{\partial \varphi} \right)_{\varphi=0} + O(\varphi) = 0. \quad (38)$$

Analogously, the scale of the spatial inhomogeneity is determined by the term at zero degrees of the field φ . However, the calculation with a spatial distribution function (9) is somewhat problematical because the underintegral expression cannot be integrated in an elementary function. That is why we have to find another method. If we suppose that the gas is ideal locally, then we have to suppose that the motion of the particles is quasiclassical [21]. Then we propose to use the particle's energy in the Boltzmann view

$$U(r) = -kT \ln \left[\frac{\rho(r)}{\rho(0)} \right] = -kT \ln e^{\varphi(r)} = -kT \varphi(r), \quad (39)$$

where the distribution (14) has been used and the field φ is the solution of equation (38); *however* we determine the radius of the cluster d as

$$U(d) = \frac{1}{2}kT + \frac{3}{5}\varepsilon_F, \quad (40)$$

where ε_F is the Fermi energy which is determined by the fixed average density of the system (A.7). The term $\frac{1}{2}kT$ is the kinetic energy of a particle caused by thermal motion and $\frac{3}{5}\varepsilon_F$ is the average particle kinetic energy in a degenerate Fermi gas (because by the Pauli exclusion principle all particles cannot be in the level $\varepsilon = 0$). The term $\frac{3}{5}\varepsilon_F$ can be understood as the repulsive 'statistical potential' [19] too. Then the assumption about local ideality of gas is written as $\frac{q^2}{\sqrt[3]{V}/N} \ll kT + \varepsilon_F$ —the average energy of interaction of two particles is less than the average kinetic energy of a particle.

Using the decomposition (38) and equations (40) and (A.3) for determination of the activity ξ_0 we can obtain an expression for the radius of the cluster for a Coulomb interacting Fermi gas:

$$\left[kT \frac{2r_q^3}{\lambda^3} \left(\frac{\partial f_{5/2}(\xi_0 e^\varphi)}{\partial \varphi} \right)_{\varphi=0} d^2 = \frac{1}{2}kT + \frac{3}{5}\varepsilon_F \right. \\ \left. \frac{\lambda^3 N}{V} = f_{3/2}(\xi_0) \right]. \quad (41)$$

Let us consider some limiting cases. In the limit of high temperature we supposed that $kT \gg \varepsilon_F$. Then calculating the first quantum correction only in the decomposition (A.3) we can obtain that

$$d^2 = \frac{V}{2r_q^3 N} + \frac{1}{2^{5/2}} \frac{\lambda^3}{r_q^3}, \quad (42)$$

where $\lambda/r_q \ll 1$. For the dimension value:

$$D^2 = \frac{kTV}{4\pi q^2 N} + \frac{1}{2^{7/2}} \frac{\hbar^3}{\pi(2\pi m)^{3/2}(kT)^{3/2}}. \quad (43)$$

The sense of the first addendum has been explained in section 3.2. The sense of the first quantum correction is explained below.

It is known that a so-called 'statistical potential' of particle interaction exists in quantum gases [19]. It is repulsion for fermions and attraction for bosons. This phenomenon is connected with the symmetry of the particle wavefunction. Hence the equilibrium size is determined by three energies—Coulomb interaction (the interaction constant is r_q or q^2), thermal kT and the above-mentioned 'statistical potential' (λ or \hbar plays the part of an interaction constant). That is, in case of Fermi statistics, the quantum energy resists compression under the action of a Coulomb force as the small addition to thermal energy for high temperatures. This explains the sign '+' for the small correction to the cluster radius. For the case of Bose statistics we can use an analogous expression with the sign '+' replaced by the sign '-' [28].

In the limit of low temperatures $kT \ll \varepsilon_F$ we obtain that

$$D^2 = 0.4 \frac{\hbar^2}{q^2 m} \left(\frac{V}{N} \right)^{1/3} \left[1 + \frac{5}{6} \frac{kT}{\varepsilon_F} \right]. \quad (44)$$

We can see that the equilibrium size is determined by two energies—the Coulomb potential and the repulsion statistical potential (the constant of interaction is \hbar). That is, the aforesaid quantum energy resists the Coulomb compression. The small correction $\frac{kT}{\varepsilon_F}$ means that the above-mentioned thermal mechanism of cluster formation (as a small addition to Fermi statistical repulsion) is valid even if $T \neq 0$.

An expression for the radius of a cluster at low temperatures can be obtained in the following way. It is not difficult to see that the equilibrium radius of a cluster is determined by the multiplier $\frac{2r_q}{\lambda^3}$ in the equation (13) after the term e^φ at $\xi \approx \xi_0$. In the case that $T \rightarrow 0$ we can suppose that the radius is determined by the above-mentioned multiplier; however, because temperature is zero, kT cannot enter into the formula. Then it is necessary to multiply this coefficient by the value $\frac{kT}{\varepsilon_F}$ which cancels the temperature. Then the cluster radius is determined as $\frac{1}{D^2} \propto \frac{r_q}{\lambda^3} \left(\frac{kT}{\varepsilon_F} \right) = \frac{\hbar^2}{q^2 m} \left(\frac{V}{N} \right)^{1/3}$, and we have the formula (44).

3.4. The condition for cluster formation and phase transition to the spatially inhomogeneous state

Since equation (11) has a soliton solution under any thermodynamic conditions, that cluster is formatted forever,

i.e. for any temperature and concentration. However, we have considered a system in the limit $N \rightarrow \infty$, $V \rightarrow \infty$ and fixed N/V . It is known that real systems have a finite volume and number of particles, where N and V are fixed. This requires small corrections in the equation of state for a system with short-range interaction between particles [20]. In the case of systems with long-range attraction the situation must be changed because the length of such an interaction is greater than the size of the system. Hence we can suppose that phase transition from an inhomogeneous state to a spatially homogeneous state can occur.

Let us write the condition of normalization for the spatial distribution (16) as

$$\int_0^V \rho(r) d^3r = \frac{N}{r_q^3} \quad \text{or} \quad \int_0^V \sigma^2 d^3r = \frac{V}{r_q^3}. \quad (45)$$

This equation may be regarded as the condition put on the solution of equation (13). We must find the field σ which minimizes the effective free energy (12) at the condition (45). Hence we must use the method of an indefinite Lagrange multiplier. The Lagrange multiplier can be found by solving a system of three equations: two determined by the asymptotics of solution (16) and the third from the condition of normalization (45). However, we can find an even simpler way.

Let us compare two effective free energies *on the saddle-point* $\tilde{\xi} = \frac{\lambda^3 N}{V}$. The first corresponds to a gas with a spatially inhomogeneous distribution which we can call the ‘condensed’ phase:

$$\begin{aligned} S_d &= 4\pi \int_0^V (\Delta^2 - 2\xi\alpha^2\sigma^2) r^2 dr + N \ln \xi \\ &= \Delta^2 \frac{V}{r_q^3} - 2\xi\alpha^2 \frac{V}{r_q^3} + N \ln \xi. \end{aligned} \quad (46)$$

The second corresponds to a spatially homogeneous distribution with $\varphi \equiv 0$, which we can call the ‘gaseous’ phase:

$$S_\infty = -\frac{V}{\lambda^3} \xi + N \ln \xi. \quad (47)$$

In the case of inhomogeneous distribution the corresponding effective free energy must be less than that for homogeneous distribution on the saddle-point:

$$\begin{aligned} S_d(\tilde{\xi}) &\leq S_\infty(\tilde{\xi}) \\ \Downarrow \\ \Delta^2 \frac{V}{r_q^3} - 2\tilde{\xi}\alpha^2 \frac{V}{r_q^3} + N \ln \tilde{\xi} &\leq -\frac{V}{\lambda^3} \tilde{\xi} + N \ln \tilde{\xi}, \end{aligned} \quad (48)$$

where the equality is reached in the point of phase transition to the ‘condensed’ phase. The inequality (48) can be reduced to:

$$\frac{r_q^3 N}{V \Delta^2} \geq 1. \quad (49)$$

Let us integrate equation (16) over all space using the normalization (45). Using the equality $\int_0^\infty \frac{x^2 dx}{\cosh^2 ax} = \frac{\pi^2}{12a^3}$ we find that $\Delta = \frac{\pi^2}{3N}$. Then for temperatures or concentrations when the inequality

$$\frac{r_q^3 N^3}{V} \geq \frac{\pi^6}{9} \quad (50)$$

is executed, the interacting gas is in a spatially inhomogeneous state. We can see that termination of the volume and number of particles means that if thermodynamic conditions do not execute the inequality (50) then a cluster cannot form. On the contrary, in the limit $N \rightarrow \infty$ and $V \rightarrow \infty$ the transition to a spatially inhomogeneous state always happens, because this condition is always satisfied in this case. In other words, phase transition to a spatially inhomogeneous state from a homogeneous state can occur in the finite system only, and the critical temperature is

$$kT_c = \frac{2\sqrt[3]{9}}{\pi} q^2 \frac{N}{\sqrt[3]{V}}, \quad (51)$$

where we have used that $r_q = \frac{2\pi q^2}{kT}$.

Let us calculate the heat which escapes at the transition: $Q = T \Delta \Omega$, where $\Delta \Omega$ is difference between the entropies of these two states. We can find the internal energies of the states with the help of the formula (A.11), $U = -\frac{\partial S}{\partial(1/kT)}$. Then we have:

$$U_d = \frac{\pi^6}{3N^2} \frac{V}{r_q^3} kT \quad (52)$$

$$U_\infty = \frac{3}{2} \frac{\xi V}{\lambda^3} kT \approx \frac{3}{2} NkT. \quad (53)$$

We can see that at such a transition the internal energy suffers a jump. The free energies of these states are determined by the effective free energies (46) and (47) on the saddle-point $\xi \approx \frac{\lambda^2 N}{V}$:

$$\begin{aligned} F_d &= kT \left(\frac{\pi^6}{9N^2} \frac{V}{r_q^3} - 2N + N \ln \xi \right) \\ &= kT \left(\frac{\pi^6}{9N^2} \frac{V}{r_q^3} - N \right) + F_\infty \end{aligned} \quad (54)$$

$$F_\infty = kT \left(-\frac{V}{\lambda^3} \xi + N \ln \xi \right) \approx kT (-N + N \ln \xi). \quad (55)$$

Knowing the internal and free energies we can find the entropies as

$$\Omega_d = \frac{U_d - F_d}{T} = k \frac{2\pi^6}{9N^2} \frac{V}{r_q^3} + kN - \frac{F_\infty}{T} \quad (56)$$

$$\Omega_\infty = \frac{U_\infty - F_\infty}{T} \approx \frac{3}{2} Nk - \frac{F_\infty}{T}. \quad (57)$$

Then the heat which escapes at the transition from a spatially homogeneous state to a spatially inhomogeneous state is

$$Q = T (\Omega_d - \Omega_\infty) = \frac{1}{2} NkT \left(\frac{4\pi^6}{9N^3} \frac{V}{r_q^3} - 1 \right) = \frac{3}{2} NkT, \quad (58)$$

where we have used the equality (50). We can see that the escaping heat is equal to the kinetic energy of the system in our approximation $\xi \approx \xi_0$. This can be interpreted as collapse of the system, when all particles stop and the system compresses to a point. It is necessary to notice that these phases are distinguished by single-particle motion: namely,

superdiffusive in the ‘condensed’ phase and ballistic in the ‘gaseous’ phase [26].

This phenomenon can be explained using simple but clear considerations. Let gas creates the field $\Delta\phi = 4\pi qn$, where $n \simeq N/V$ and $V \sim R_{\max}^3$. If we suppose that the field $\phi = 0$ is on the border of the system, then the energy of a particle in the center (the depth of the potential well) is

$$U_0 = q\phi \sim -q^2 n R_{\max}^2. \quad (59)$$

The thermal energy of a particle is $\sim kT$. If the thermal energy of a particle is less than its maximum energy of interaction (in the center of the system), then the gas has a negative pressure and tries to compress to a point. On the contrary, if the thermal energy of a particle is more than its maximum interaction energy (by module), then gas has a positive pressure and tries to widen to infinity. When gas is compressing, its thermal energy is enlarging, since the total energy of the system is constant. This establishes the equilibrium spatial distribution (16).

The condition of cluster formation is determined by correlation between above-mentioned energies:

$$q^2 n R_{\max}^2 \sim kT \Rightarrow \frac{q^2 N}{kT R_{\max}} \sim 1. \quad (60)$$

If $N \rightarrow \infty$ and $V \rightarrow \infty$, then $U_0 \rightarrow \infty$ —this means that thermal energy is always less than the energy of interaction in the center of the system (any point can be chosen as the center of the system in this case), hence clusters exist at any temperature in such a gas.

The same reasoning can be applied to a degenerate Fermi gas, where Fermi energy ε_F supplies part of particle’s kinetic energy. We can write that

$$q^2 n R_{\max}^2 \sim \varepsilon_F \Rightarrow \frac{q^2 m N^{1/3} R_{\max}}{\hbar^2} \sim 1. \quad (61)$$

Hence the phase transition to the spatially inhomogeneous state occurs in a degenerate Fermi gas with attracting Coulomb interaction. We can obtain the same equations from Jeans criterion $D \sim R_{\max}$ with help equations (25) and (44).

We can see that the phase transition from a spatially homogeneous state to a spatially inhomogeneous state in systems with long-range interacting particles occurs in finite systems only. In an infinite system (when number of particles $N \rightarrow \infty$ and volume $V \rightarrow \infty$ with fixed N/V) clusters exist at any temperature and the phase transition does not occur.

4. Conclusion

Based on a statistical approach [3, 4, 1, 2], we have demonstrated a nonperturbative calculation of the partition function and solved the system of particles with Boltzmann and Fermi statistics with long-range attraction of the Coulomb type. That is, we obtained the free energy for such a system and considered phase transition between a spatially homogeneous state and a spatially inhomogeneous state. We obtained the expression for the equilibrium radius of a cluster

and the conditions of formation of a spatially inhomogeneous distribution in the system for all temperatures. However, we can find exact results in the Boltzmann limit with first quantum corrections. For all temperatures we proposed an approach which gives the possibility of finding the approximate radius of the cluster, or to evaluate one. All results were obtained for the condition that the average energy of interaction of two particles is much less than their average kinetic energy kT or Fermi energy ε_F : $\frac{q^2}{\sqrt[3]{V/N}} \ll kT$ or $\frac{q^2}{\sqrt[3]{V/N}} \ll \varepsilon_F$ —the ‘local ideality’ approach [24].

The long-range attraction of particles results in the formation of a cluster of finite size, as the initial homogeneous state is unstable in the limit $N \rightarrow \infty$, $V \rightarrow \infty$ with fixed N/V . The equation for the spatial distribution function has been obtained. However, it is simplified in the following limiting cases only.

In the case of large temperature (when $\lambda \ll \sqrt[3]{v}$) the radius of a cluster is determined by the balance of two energies: the energy of attractive interaction which tries to compress the gas, and the thermal energy which creates a positive pressure. This mechanism created an equilibrium distribution in the system (16), in particular, with increasing temperature the size of the cluster increases and its central density decreases. In the case of a degenerate system the radius of the cluster is also determined by the balance of two energies: the energy of interaction which tries to compress the gas, and the repulsive Fermi statistical potential creating a positive pressure. Overall the equilibrium cluster size is determined by three energies: the energy of interaction and the thermal and repulsive ‘statistical potential’. The size of the cluster approaches the equilibrium size asymptotically during its formation. The state of the system with a spatially inhomogeneous distribution corresponding to a cluster of the equilibrium size (43) or (44) is stable if we assume that the ‘local ideality’ approach is true.

It is interesting to note that if we are assuming the constant interaction is $q^2 = Gm^2$, where m is the particle mass and G the gravitational constant, then the cluster radii (43), (44) are Jeans lengths in a self-gravitational system. However, if we are assuming the constant interaction is the electrical constant $q^2 = (Ze)^2$, then the above-mentioned cluster radii are Debye–Hueckel and Thomas–Fermi radii of screening in a plasma.

The situation changes in the case of large but finite size of the system and number of particles in it. The temperature and average concentration in the system need to satisfy the inequality (50) in the Boltzmann limit and (61) in the degenerate case for existence of the cluster. For decreasing temperature, when it reaches the value which satisfies the aforesaid inequalities (50), (61), phase transition to a spatially inhomogeneous distribution occurs. The radius of a cluster plays the part of correlation length for such a phase transition. The internal energy suffers a jump and heat escapes at the point of transition. Such behavior of the system with long-range interacting particles occurs because the length of the interaction is comparable with the size of the system. In other words, the phase transition from the spatially homogeneous state to the spatially inhomogeneous state in the system with

long-range interacting particles occurs in a finite system only. In an infinite system (when the number of particles $N \rightarrow \infty$ and volume $V \rightarrow \infty$ with fixed N/V) clusters exist at any temperature and phase transition does not occur.

Appendix. The effective free energies for ideal quantum gases

Let us obtain some expressions used in this paper for ideal Fermi and Bose gases. Thus we shall demonstrate the correctness and greater rationality of our approach as compared with the traditional method [19].

In the case of an ideal gas $\varphi = \psi = 0$. Then the effective free energy (3) for the system is

$$S = -\frac{1}{\omega} \int dV \int 4\pi p^2 dp \ln(1 + \xi \exp(-\beta \varepsilon_p)) + N \ln \xi$$

$$= -\frac{V}{\lambda^3} f_{5/2}(\xi) + N \ln \xi, \quad (\text{A.1})$$

where

$$f_{5/2}(\xi) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \ln(1 + \xi e^{-x^2})$$

$$= \sum_{l=1}^\infty (-1)^{l+1} \frac{\xi^l}{l^{5/2}} \quad (\text{A.2})$$

is the special Fermi function [19, 20]; $\varepsilon_p = p^2/2m$ is the kinetic energy of a particle; and $\lambda = \sqrt{\frac{\beta \hbar^2}{2\pi m}}$ is the thermal wavelength of a particle. Then the equation for the saddle-point is as follows:

$$\frac{1}{v} = \frac{1}{\lambda^3} f_{3/2}(\xi), \quad (\text{A.3})$$

where $f_{3/2}(\xi) = \xi \frac{\partial f_{5/2}(\xi)}{\partial \xi}$, $v \equiv \frac{V}{N}$.

The partition function (2) for this case:

$$Z_N = \exp \left[\frac{V}{\lambda^3} f_{5/2}(\xi) - N \ln \xi \right], \quad (\text{A.4})$$

where ξ is determined from equation (A.3). We can find any thermodynamic functions knowing the partition function. Let us consider the case $\xi \rightarrow 0$ corresponding to high temperature (the Boltzmann limit), then equations (A.3), (A.4) are reduced to

$$Z_N = \exp \left[\frac{V}{\lambda^3} \xi - N \ln \xi \right] \approx \frac{V^N}{N!} \left(\frac{mkT}{2\pi \hbar^2} \right)^{\frac{3}{2}N} \quad (\text{A.5})$$

$$\frac{1}{v} = \frac{\xi}{\lambda^3}. \quad (\text{A.6})$$

Let us consider the case $T \rightarrow 0$ corresponding to a degenerate Fermi gas. Then the special Fermi functions are [19]

$$f_{3/2} = \frac{4}{3\sqrt{\pi}} \left[(\ln \xi)^{3/2} + \frac{\pi^2}{8} (\ln \xi)^{-1/2} \right] + O(\xi^{-1})$$

$$f_{5/2} = \frac{4}{3\sqrt{\pi}} \left[\frac{2}{5} (\ln \xi)^{5/2} + \frac{\pi^2}{4} (\ln \xi)^{1/2} \right] + O(\xi^{-1}). \quad (\text{A.7})$$

In this case, the expression (A.3) is reduced to

$$\frac{\lambda^3}{v} \approx \frac{4}{3\sqrt{\pi}} (\ln \xi)^{3/2} \Rightarrow \xi \approx e^{\beta \varepsilon_F}, \quad (\text{A.8})$$

where

$$\varepsilon_F \equiv \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{v} \right)^{\frac{2}{3}} \quad (\text{A.9})$$

is the Fermi energy.

We can find analogous expressions for a Bose gas. It is necessary to proceed from the effective free energy as

$$S = \frac{1}{\omega} \int dV \int 4\pi p^2 dp \ln(1 - \xi \exp(-\beta \varepsilon_p)) + N \ln \xi$$

$$= -\frac{V}{\lambda^3} g_{5/2}(\xi) + \ln(1 - \xi) + N \ln \xi, \quad (\text{A.10})$$

where $g_{5/2}(\xi) = -\frac{4}{\sqrt{\pi}} \int_0^\infty dx x^2 \ln(1 - \xi e^{-x^2}) = \sum_{l=1}^\infty \frac{\xi^l}{l^{5/2}}$ is a special Bose function [19, 20]. It is clear that activity is always $\xi < 1$ unlike a Fermi system; $\ln(1 - \xi)$ is the effective free energy for a condensed phase (the addendum with $\mathbf{p} = 0$ is as important as the rest of the sum when $\xi \rightarrow 1$).

Let us find the internal energy U of the system proceeding from the arranged analogy between thermodynamic theory in our representation and field theory. In order to do this we will use the correlation $-H = \partial S_{\text{mech}}/\partial t$ and determine conformities $H \longleftrightarrow U$, $S_{\text{mech}} \longleftrightarrow S_{\text{term}}$, $t \longleftrightarrow 1/kT$, where H is Hamilton's function of the system, S_{mech} and S_{term} are the action for mechanics and effective free energy for thermodynamics (3) systems, respectively, t is time and $1/kT$ is reverse temperature. Then, with the help of expression (8), we have

$$U = -\frac{\partial S}{\partial (1/kT)} = \frac{3}{2} \frac{VkT}{\lambda^3} f_{5/2}(\xi) = \frac{3}{2} PV. \quad (\text{A.11})$$

Thus the expressions for the thermodynamic function of Fermi and Bose gases obtained with our method coincide with those obtained in the usual way [19, 20], confirming the correctness of the proposed approach.

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